Application of mathematical radon transformations to describe acoustic wave propagation in continuum environments

Olena Azarenko * 1 A; Yulia Honcharenko 2 B; Mykhailo Divizinyuk 3 C;
Oleg Shevchenko 4 D; Viktoriia Mogylevska 5 F

***Corresponding author:** [1] Dr, Professor, Deputy Head, e-mail: e-mail: azarenko_ev@ukr.net, ORCID: 0000-0003-2927-5545
[2] Dr of Physico-Mathematical Sciences, Associate Professor, Professor of Department, e-mail: vup@e-u.in.ua, ORCID: 0000-0003-2045-0263
[3] Dr of Physico-Mathematical Sciences, professor, e-mail: divizinyuk@ukr.net, ORCID: 0000-0002-5657-2302
[4] PhD in Economics, Associate Professor, e-mail: oshevchenko1@hotmail.com, ORCID: 0000-0003-0431-7307
[5] leading researcher, e-mail: geka090582@ukr.net, ORCID: 0000-0002-3939-7717

[A] Research laboratory-experimental center "BRAND TRADE", Kyiv, Ukraine
[B] European University, Kyiv, Ukraine
[C] Institute of Environmental Geochemistry of the National Academy of Sciences of Ukraine, Kyiv, Ukraine
[D] Kyiv National University of Technologies and Design, Kyiv, Ukraine
[F] National Defence University of Ukraine named Ivan Chernyakhovskiy, Kyiv, Ukraine

Received: June 12, 2022 | Revised: June 22, 2022 | Accepted: June 30, 2022

DOI: 10.5281/zenodo.7699471

Abstract

The paper investigates the application of mathematical Radon transformations to describe the propagation of acoustic waves in continuum environments. It is shown that these transformations make it possible to analytically take into account many factors that affect the propagation of acoustic waves. It also takes into account the properties of objects that reflect it. This version of the description can be used as a generalized mathematical model, which is considered as a spherically symmetric function from the distance to the source of coherent waves, described by the Radon transform.

Key words: acoustic wave, continuum environments, Radon transform, devices, mathematical model.

Introduction

Solving urgent applied problems of detecting anthropogenic impurities in water and air [1, 2], searching for underwater objects and surveying the seabed [3, 4], preventing emergencies and protecting the natural environment [5-7], studying the state of organs of living people [8, 9] and many other tasks are provided by the use of various acoustic devices [10, 11]. The creation of all these devices is based on the laws of propagation of elastic waves in continuum environments [12, 13]. Undoubtedly, a ship's sonar echo sounder and an ultrasonic scanner installed in a burglar alarm are acoustic devices. They use the same echolocation method [14]. Simultaneously, their design and practical use are based on the fundamental difference between the studied environments, the spatio-temporal scales of detected acoustic anomalies, and the frequency range of elastic vibrations used. Naturally, the sound propagation models used to develop and construct these devices differ from each other [2, 4, 10, 12, 15]. For these reasons, they use various elemental components, both for generating acoustic waves and for processing echo signals [1, 8, 9, 11, 13, 16]. Creation of a universal or generalized model of propagation of acoustic waves in continuum environments will allow, on the one hand, to unify the software that ensures the operation of various acoustic devices. On the other hand, it will allow using the same microprocessor components for the design and production of acoustic instruments and devices for various purposes.

Based on the above, the purpose of this paper is to investigate the application of mathematical Radon transformations to describe the propagation of acoustic waves in continuum environments.

To achieve this goal, it is necessary to solve the following tasks. Initially, analyze the physical laws of the propagation of acoustic waves in continuum environments. Then consider the mathematical transformations of Radon to describe the propagation of acoustic waves in continuum environments.

Results and discussion

Physical laws of acoustic wave propagation in continuum environments

In continuum environments with distributed parameters, it is possible to excite compression and rarefaction oscillations that propagate at a certain speed. In liquids characterized only by bulk elasticity, longitudinal acoustic waves can arise, in which the direction of oscillation of the particles of the environment coincides with the direction of wave propagation [2, 3, 10, 12, 13].

Let's denote the elementary volume of the elastic environment V_0, its density ρ_0, and the static pressure P_0, which were in the stationary state of the environment before the excitation of sound vibrations.

Exciting oscillations, the applied external force causes displacement of particles of the environment and changes the volume, density and pressure to the values V, ρ, P. Then the relative changes in volume (volumetric deformation) and density (compaction) will be respectively equal to:

$$\Delta V = \frac{V - V_0}{V_0} = \frac{\delta V}{V_0}; \qquad \Delta\rho = \frac{\rho - \rho_0}{\rho_0} = \frac{\delta\rho}{\rho_0}. \tag{1}$$

These values can be both positive and negative.

For small deformations, when $\Delta V << 1$ и $\Delta\rho << 1$, the mass conservation law ($\rho V = \rho_0 V_0 = const$), is taken into account, and from relations (1) it follows that:

$$\Delta\rho = -\Delta V. \tag{2}$$

That is, at small deformations, the compaction is equal to the volumetric deformation and is opposite in sign to it. A change in the density of the volume element of the environment leads to a change in pressure. It will be composed of the initial static pressure and excess dynamic (acoustic pressure):

$$P = P_0 + p. \tag{3}$$

Note that in problems of measuring the speed of sound in an aqueous medium, the acoustic pressure is much less than the static one. In general, pressure in a liquid is a function of density and temperature.

In low-amplitude acoustic waves ($p << P$), the alternation of compression and rarefaction occurs so quickly that heat transfer between these volumes does not occur during the oscillation period, therefore the sound wave propagation process is adiabatic, with a random pressure equal to the sum of static and acoustic pressure , in a continuum environment will be a function depending on the density, that is:

$$P = f(\rho). \tag{4}$$

Consequently, an acoustic wave is an adiabatic process of successive transfer of compressions and rarefactions from one local part of the environment to another, which is characterized by excess (acoustic) pressure as a single-valued function of density.

The peculiarities of acoustic wave propagation for certain conditions are defined in terms of specific physical models. A mathematical description of each physical model will provide its subsequent algorithms and software implementation. This allows analytical calculations of the expected detection ranges of acoustic objects and visualization of a qualitative change in the state of a continuum environment along the propagation of acoustic waves [8, 9, 11, 15].

Waves in continuum environments propagate according to cylindrical and spherical laws. If the dimensions of the wave sources and the distances over which they propagate are commensurate, the cylindrical law works.

If the distance over which the wave propagates is much greater than the dimensions of its source, then the source is considered to be a point source, and the propagation of waves occurs according to a spherical law. In the first case, the wave front expands like the walls of a cylinder, and the intensity of the wave decreases in proportion to the distance. In the second case, the wave front expands like the surface of a sphere, and the intensity of the wave decreases in proportion to the square of the distance.

As a result of relaxation processes occurring in continuum environments, waves are attenuated by a certain amount, the specific value of which depends both on the frequency of the considered waves and on some physical parameters characterizing the continuum medium in which they propagate.

The intensity of the propagating wave can be increased by a value determined by the directional action of the wave source, which is taken into account by the concentration factor of the wave source. Similarly, the impact of interference will be reduced due to the directional properties of the wave receiver, which will be determined by their concentration factors.

In the theory of wave propagation, the concept of a beam is accepted – the direction orthogonal to the front of a propagating wave. The use of rays makes it possible to visually display the configurations of wave fields formed by directional wave sources. In homogeneous environments, the rays are straight, and in inhomogeneous, the rays are bent, refract (turn) in the direction of decreasing the speed of wave propagation. The degree of this curvature (refraction) will be the greater, the higher the values of the sound velocity gradient in the environment. The distribution of sound velocity values in continuum media is commonly called the sound propagation velocity field. It is determined by the structure of the environment under consideration. In practice, it is customary to describe it by two coordinate dependencies - horizontal and vertical distributions of the speed of sound propagation. In turn, this makes it possible to construct ray patterns and solve various applied problems.

Application of mathematical Radon transformations to describe acoustic wave propagation in in continuum environments

Strictly speaking, a ray transform P maps a function defined in an *n*-dimensional Euclidean space R^n, into a set of linear integrals [8, 9, 16]. In other words, if the unit vector θ belongs to the unit sphere S^{n-1}, i.e. $\theta \in S^{n-1}$, and *x* belongs to the Euclidean space R^n, i.e. $x \in R^n$, then

$$Pf(\theta, x) = \int_{-\infty}^{+\infty} f(x + t\theta) dt \tag{5}$$

is the integral of the function f, which belongs to the Schwartz space $f \in \Psi(R^n)$, along the straight line passing through the point *x* in the direction θ.

The value $Pf(\theta, x)$ does not change when the *x* point is moved in the direction θ. As a rule,

x is chosen from an orthogonal subspace $\theta^{\perp}$, so Pf it is given on the tangent bundle of the sphere S^{n-1}.

The elements of the ray pattern also make it possible to calculate the regularities of the decay of the acoustic field in a continuum environment and the values of its focusing coefficients (increase or decrease in the intensity of the wave due to the features of the environment, for example, focusing and concentration of sound by an underwater sound channel, attenuation of the intensity of acoustic waves by a shock layer, and others).

The formation of ray constructions is also influenced by the presence of water fronts and currents, the depth of the sea and the topography of the bottom, waves of the water surface, and other factors.

For a mathematical description of the physical processes of detecting acoustic objects, one can use a mathematical apparatus called the Radon transform. It allows you to take into account the influence of all the above factors on the propagation of waves.

Let $f \in \Phi$, then for $m = 0, \quad 1, \quad \ldots$ fair

$$\int_{R^1} s^m \mathbf{R}_\theta f(s) ds = p_m(\theta), \tag{6}$$

$$\int_{\theta^{\perp}} (x \cdot y)^m \mathbf{P}_\theta f(x) dx = q_m(y), \quad y \perp \theta, \tag{7}$$

Where R_θ – the Radon transform;

P_θ – ray transform describing the ray pattern of the acoustic field;

$p_m, \quad q_m$ – homogeneous polynomials of degree m, and q_m does not depend on θ.

Putting $x = s\theta + y$, we get

$$\int_{R^1} s^m \mathbf{R}_\theta f(s) ds = \int_{R^1} s^m \int_{\theta^{\perp}} f(s\theta + y) dy ds = \int_{R^n} (x \cdot \theta)^m f(x) dx. \tag{8}$$

Expression (8) is a homogeneous polynomial of degree m from θ. Arguing similarly, by changing variables $z = x + t\theta$, we obtain for $y \perp \theta$ a homogeneous polynomial of degree m from y, which does not depend on θ.

$$\int_{\theta^{\perp}} (x \cdot y)^m \mathbf{P}_\theta f(x) dx = \int_{\theta^{\perp}} (x \cdot y)^m \int_{R^1} f(x + t\theta) dt dx = \int_{R^n} (z \cdot y)^m f(z) dz. \tag{9}$$

From properties (6) and (7) of the Radon transform, the following assertion is true.

If $f \in C_0^{\infty}(\Omega^n)$, then *Rf* can be expanded in a series in terms of functions $C_l^\lambda Y_{kj}$,

where C_l^λ – are the Gegenbauer polynomials,

Y_{kj} – are spherical harmonics forming a complete orthogonal system in $L_2\left(Z, \quad (1 - s^2)^{\frac{\lambda - 1}{2}}\right)$.

This decomposition looks like:

$$\mathbf{R}f(\theta, s) = (1 - s^2)^{\frac{\lambda - 1}{2}} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j} c_{lkj} C_l^\lambda(s) Y_{kj}(\theta) \tag{10}$$

where j runs through all $N(n, k)$ spherical harmonics of degree k.

Let $g \in \Phi(T)$ and $g(\theta, x) = 0$ at $|x| \geq a$. Let at $m = 0, \ 1, \ \ldots$

$$\int_{\theta^\perp} (x \cdot y)^m g(\theta, x) dx = q_m(y), y \perp \theta \tag{11}$$

is a homogeneous polynomial of degree m from y, independent of θ. Then there is a function $f \in \Phi(R^n)$, for which *g* = *Pf* and $f(x) = 0$ for $|x| \geq a$.

Let us show that, then *g* = *Pf*, can be expressed in terms *Pf*(ω, s) of *Pf*. Therefore, consider for $\theta \perp \omega$ the function

$$h_\theta(\omega, s) = \int_{\substack{\theta^\perp \\ x \cdot \omega = s}} g(\theta, x) dx, \tag{12}$$

Let us show that h_θ does not depend on the choice of θ and $h_\theta = Rf$ for some function $f \in \Phi(R^n)$ supported in the domain $|x| \leq a$.

Note that

$$\int_{R^1} s^m h_\theta(\omega, s) ds = \int_{R^1} s^m \int_{\substack{\theta^\perp \\ x \cdot \omega = s}} g(\theta, x) dx ds = \int_{\theta^\perp} (x \cdot \omega)^m g(\theta, x) dx = q_m(\omega), \tag{13}$$

where q_m – polynomial.

Since it is independent q_m of θ, the integral on the left side of (13) is also independent of θ. The function $h_\theta(\omega, s) = 0$ for $|s| \geq a$, since $g(\theta, x) = 0$ for $|x| \geq a$. The density of polynomials in $L_2(-a, a)$ implies independence h_θ of θ, and, in accordance with (13), the function $h = h_\theta$ satisfies conditions (9). Hence, *h* = *Pf* for some function $f \in \Phi\left(R^n\right)$ supported in the domain $|x| \leq a$.

Let's make sure that *g* = *Pf* at $n > 2$. To do this, we show that for a fixed θ, the integrals of the functions g and *Pf* over arbitrary planes in $\theta^\perp$ coincide.

Consider the plane $\{x \in \theta^\perp : x \cdot \omega = s\}$, where $\omega \in \theta^\perp$. The integral of the function g over this plane is $h(\omega, s)$, and

$$\int_{\substack{\theta^\perp \\ x \cdot \omega = s}} \mathbf{P} f(\theta, x) dx = \int_{\substack{\theta^\perp \\ x \cdot \omega = s}} \int_{-\infty}^{\infty} f(x + t\theta) dt dx = \int_{\theta^\perp \cap \omega^\perp} \int_{-\infty}^{\infty} f(s\omega + y + t\theta) dt dx = \tag{14}$$

$$= \int_{\omega^\perp} f(s\omega + z) dz = \mathbf{R} f(\omega, s).$$

Since *h* = *Rf* the integral of the functions g and *Pf* are equal to each other. This means that the Radon transforms of the function $g(\theta, \cdot)$ and *Pf*(0,) on $\theta^\perp$ coincide.

Then $g \in \Phi(T)$ and

$\int_{\theta^\perp} (x \cdot y)^m g(\theta, x) dx = u(\theta) \int_{\theta^\perp} (x \cdot y)^m h(|x|) dx = u(\theta) \int_0^\infty s^{n-2+m} h(s) ds \int_{s^{n-1} \cap \theta^\perp} (\omega \cdot y)^m d\omega = 0,$

where $x = s\omega \in \theta^\perp$.

This implies that q satisfies the compatibility conditions (11).

A special case when $n > 3$, and $\omega \perp \theta$, then

$$\int\limits_{x\perp\theta, x\cdot\omega=s} g(\theta,x)dx = u(\theta) \quad \int\limits_{x\perp\theta, x\cdot\omega=s} h(|x|)dx = u(\theta)\mathbf{R}h(\omega,s) \tag{15}$$

where h – spherically symmetric function of x,
$R-(n-1)$ – dimensional Radon transform.

The considered properties of expression (14) make it possible to define it as a generalized case of the mathematical description of wave propagation, which analytically takes into account many factors that affect its propagation. This version of the description can be used as a generalized mathematical model, which is considered as a spherically symmetric function from the distance to the source of coherent waves, described by the $(n-1)$ – dimensional Radon transform.

Conclusions

Thus, the mathematical apparatus of Radon can be used for various cases, the description of acoustic fields in continuum environments. It allows one to analytically take into account many factors that affect the propagation of acoustic waves. It also takes into account the properties of objects that reflect it. This version of the description can be used as a generalized mathematical model, which is considered as a spherically symmetric function from the distance to the source of coherent waves, described by the Radon transform. The subsequent algorithmization and software implementation of this model will allow analytical calculations of the expected detection ranges of acoustic objects and visualization of a qualitative change in the state of a continuum environments along the propagation of acoustic waves. This will also allow the use of the same microprocessor components for the design and manufacture of acoustic instruments and devices for various purposes.

References

[1.] Azarenko Y.V, Goncharenko Y.Y., Divzinyuk M.M. (2012). Modeli rasprostraneniya antropogennoy primesi v Chernom more. Monografiya. Sevastopol': SNUYAEiP, 93 p. ISBN-978-966-8962-05-9. [in Ukrainian].

[2.] Tatarskiy V.I. (1959). Teoriya fluktuatsionnykh yavleniy pri rasprostranenii voln v turbulentnoy atmosphere.Monografiya. Moscow: Akademiya Nauk, 282 p. [in Russian].

[3.] Azarenko Ye.V. (2003). Akusticheskoye obnaruzheniye ob'yektov v vodnoy srede. Sevastopol': Gos. okeanarium, 71 p. ISBN-966-96-287-3-3. [in Ukrainian].

[4.] Yevtyutov A.P., V.B. Mit'ko. (1988). Primery inzhenernykh raschetov v gidroakustike. Leningrad: Sudostroyeniye. 288 p. [in Russian]

[5.] Divizinyuk M.M., Yeremenko S.A., Lyeftyerov O.A., Prus'kyy A.V., Strilets V.V., Strilets V.M., Shevchenko R.I. (2022). Teoretychny zasady paradyhmy "tsyvil'nyy zakhyst". Monohrafiya. Kyiv.: TOV "AZYMUT-PRINT". 335 p. ISBN 978-617-8015-20-6. [in Ukrainian].

[6.] Kucherova H., Didenko A., Honcharenko Y., Kravets 0., Uchitel A. (2020). Scenario Forecasting Information Transparency of Subjects' under Uncertainty and Development of the Knowledge Economy. CEUR Workshop Proceedings. Cep. "M3E2-MLPEED 2020 – Proceedings of the Selected Papers of the Special Edition of International Conference on Monitoring, Modeling and Management of Emergent Economy". pp. 81-106.

[7.] Honcharenko Y., Martyniuk O., Open'ko P., Radko O. (2020). The Method of Proactive Risk Assessment for Flight Safety Based on the Rate of Dangerous Events. *Advances in Military Technology*. T. 15. № 2. pp. 365-377.

[8.] Louis A.K. (1981) Analytische Methoden in der Computer Tomographie. – Habilitationsschrift: Fachebereich Mathematik der Universitet Munster.220 p.

[9.] Tikhonov A.N., Arsenin V.Y, Timonov A.A. (1987). Matematicheskiye zadachi komp'yuternoy tomografii. Moscwa: Nauka, 161 p.

[10.] Divizinyuk M.M. (1998). Akusticheskiye polya Chernogo moray.Monografiya. Sevastopol': Gos. Okeanarium, 352 p. ISBN-966-06-0619-4

[11.] Bulatsyk O.O., Voytovych M.M., Katsenelenbaum B.Z. (2012). Fazovi optymizatsiyni zadachi zastosuvannya v teoriyi khvyl'ovykh poliv. dr. Kyiv: Naukova dumka, 320 p.

[12.] Doronin Y.P. (1978). Fizika okeana. Leningrad: Gidrometeoizdat, 286 p.

[13.] Grinchenko V.T., Vovk I.V., Matsipura V.T. (2013). Volnovyye zadachi akustiki, Kyiv: Interservis, 572 p.

[14.] Mit'ko V.B. Yevtyutov, A.P. Gushchin S.Ye. (1982). Gidroakusticheskiye sredstva svyazi i nablyudeniya. Monografiya Leningrad: Sudostroyeniye,. 197 s.

[15.] Honcharenko Y.Y. (2014). Fyzycheskaya model' s"ema rechevoy ynformatsyy /. Zbirnyk naukovykh prats' "Systemy obrobky informatsiyi". Kharkiv: Kharkivs'kyy universytet Povitryanykh Syl imeni Ivana Kozheduba, Vyp. 7 (123). pp. 18-23.

[16.] Natterer F. (1984) Some nonstandard Radon problems. Teubuer: Application of Mathematics in Technology, 343 p.